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A METHOD FOR DEVELOPING $\cos^n\theta$ AND $\sin^n\theta$.

By M. C. STEVENS, A. M., Department of Mathematics, Purdue University, Lafayette, Indiana.

De Morgan in his Calculus gives a method for expanding $\cos^n\theta$ and $\sin^n\theta$ when n is an integer which I have not noticed in any of our American works on that subject. As it leads to an easy method for integrating such expressions as

$$\int\cos^n\theta d\theta,\quad \int\sin^n\theta d\theta,$$

etc. I have thought it might be of interest to some of the readers of the MONTHLY.

The method is as follows :

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}.\quad \text{Let } e^{i\theta}=x,\text{ then } e^{-i\theta}=\frac{1}{x},\text{ and } \cos\theta=\tfrac{1}{2}\left(x+\frac{1}{x}\right)\dots\dots\dots\dots(1)$$

$$e^{ni\theta}=x^n,\text{ then } e^{-ni\theta}=\frac{1}{x^n},\ \cos n\theta=\tfrac{1}{2}\left(x^n+\frac{1}{x^n}\right).$$

$$\text{Then from (1) } \cos^n\theta=\frac{1}{2^{n-1}}\left[\tfrac{1}{2}\left(x^n+\frac{1}{x^n}\right)+n\tfrac{1}{2}\left(x^{n-2}+\frac{1}{x^{n-2}}\right)\right.$$

$$\left.+\frac{n(n-1)}{2}\tfrac{1}{2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right)\dots\dots\right.$$

$$=\frac{1}{2^{n-1}}\left[\cos n\theta+n\cos(n-2)\theta+\frac{n(n-1)}{2}\cos(n-4)\theta+\ \dots\dots\right.$$

If n be an even number$=2m$, there will be $2m+1$ terms in the development, which will give m cosines, namely, those of $2m\theta$, $2(m-1)\theta\dots\dots$down to 2θ, and an additional term which will not contain θ, the value of which is $\dfrac{2m(2m-1)\dots\dots m+1}{\lfloor m}$. But if n be odd, and$=2m+1$, then there are $2m+2$ terms giving $m+1$ cosines, namely, those of $(2m+1)\theta$, $(2m-1)\theta\dots\dots\dots$ down to θ, with no middle term. Thus we have

$$\cos^6\theta=\tfrac{1}{32}(\cos6\theta+6\cos4\theta+15\cos2\theta+10).$$

Whence the integral of $\cos^6\theta d\theta=\frac{1}{192}\sin6\theta+\frac{3}{64}\sin4\theta+\frac{15}{64}\sin2\theta+\frac{5}{16}\theta$.

Also $\cos^7\theta=\frac{1}{64}(\cos7\theta+7\cos5\theta+21\cos3\theta+35\cos\theta)$.

Whence $\int\cos^7\theta d\theta=\frac{1}{448}\sin7\theta+\frac{7}{320}\sin5\theta+\frac{7}{64}\sin3\theta+\frac{35}{64}\sin\theta$.

The advantage of this method will be still more apparent by integrating* $\cos^3 3\theta \cos\theta d\theta$. Here $\cos^3 3\theta = \frac{1}{8}(x^3+x^{-3})^3 = \frac{1}{8}(x^9+x^{-9}) + 3(x^3+x^{-3})$.

Multiplying this by $\frac{1}{2}(x+x^{-1})$ we at once have

$$\cos^3 3\theta \cos\theta = \tfrac{1}{8}\cos 10\theta + \tfrac{1}{8}\cos 8\theta + \tfrac{3}{8}\cos 4\theta + \tfrac{3}{8}\cos 2\theta.$$

Whence $\int \cos^3 3\theta \cos\theta d\theta = \frac{1}{80}\sin 10\theta + \frac{1}{64}\sin 8\theta + \frac{3}{32}\sin 4\theta + \frac{3}{16}\sin 2\theta$.

It will be noticed that this form is well adapted for substituting values as limits of integration. For instance if the inferior limit be 0, and the superior limit $\frac{1}{6}\pi$ then $\frac{1}{80}\sin\frac{10}{6}\pi = -\frac{1}{160}\sqrt{3}$; $\frac{1}{64}\sin\frac{8}{6}\pi = -\frac{1}{128}\sqrt{3}$; $\frac{3}{32}\sin\frac{4}{6}\pi = \frac{3}{64}\sqrt{3}$; $\frac{3}{16}\sin 2\theta = \frac{3}{32}\sqrt{3}$.

$$\therefore \int_0^{\frac{1}{6}\pi} \cos^3 3\theta \cos\theta d\theta = \tfrac{81}{640}\sqrt{3}.$$

The reader will have no difficulty in applying the same method to develop $\sin^n\theta$ and then for integrating $\sin^n\theta d\theta$.

It will be observed that when we put $\cos\theta = \frac{1}{2}(x+\dfrac{1}{x})$ we do not escape the impossible; for this is as much an impossible form as $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$ for $x+\dfrac{1}{x}$ can never be *less* than 2, and $2\cos\theta$ can never be *greater* than 2.

CONCERNING CONICS THROUGH FOUR POINTS.

By EDGAR H. JOHNSON, Professor of Mathematics, Emory College, Oxford, Georgia.

The equation of the conic through a_1b_1, a_2b_2, a_3b_3, a_4b_4, and a fifth point x_1y_1 is

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1 & x_1 & y_1 & 1 \\ a_1^2 & a_1b_1 & b_1^2 & a_1 & b_1 & 1 \\ a_2^2 & a_2b_2 & b_2^2 & a_2 & b_2 & 1 \\ a_3^2 & a_3b_3 & b_3^2 & a_3 & b_3 & 1 \\ a_4^2 & a_4b_4 & b_4^2 & a_4 & b_4 & 1 \end{vmatrix} = 0,$$

or $Ax^2+2Bxy+Cy^2+2Fx+2Gy+H=0$, where the coefficients A, B, C, ... are of the second degree in x_1 and y_1. The conic is an ellipse, parabola, or hy-

*Professor Waldo first called my attention to this easy method for integrating this particular expression.